Lecture 21

Our last examples showed the the System-Rank Theorem did indeed use the rank of a matrix to predict whether or not a system was consistent, and if it was consistent, how many parameters the solution would have. Of course, we already knew the answers to these problems as we had previously solved those systems. Here we look at another example of using the System-Rank Theorem to predict how many solutions a system will have based on the values of the coefficients in the system. In this situation, we are not concerned with what the solutions are, but simply if solutions exist and how many solutions there are.

Example 21.1. For which values of $k, \ell \in \mathbb{R}$ does the system

$$\begin{array}{rcrcl} 2x_1 & + & 6x_2 & = & 5 \\ 4x_1 & + & (k+15)x_2 & = & \ell+8 \end{array}$$

have no solutions? A unique solution? Infinitely many solutions?

Solution. Let

$$A = \begin{bmatrix} 2 & 6 \\ 4 & k+15 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 5 \\ \ell+8 \end{bmatrix}.$$

We carry $[\,A \mid \vec{b}\,]$ to REF.

$$\left[\begin{array}{cc|c} 2 & 6 & 5 \\ 4 & k+15 & \ell+8 \end{array}\right] \underset{R_2-2R_1}{\longrightarrow} \left[\begin{array}{cc|c} 2 & 6 & 5 \\ 0 & k+3 & \ell-2 \end{array}\right]$$

If $k+3 \neq 0$, that is if $k \neq -3$, then $\operatorname{rank}(A) = 2 = \operatorname{rank}\big([\,A \mid \vec{b}\,]\big)$ so the system is consistent with $2 - \operatorname{rank}(A) = 2 - 2 = 0$ parameters. Hence we obtain a unique solution. If $k+3 = 0$, that is if $k = -3$, then we have

$$\left[\begin{array}{cc|c} 2 & 6 & 5 \\ 0 & k+3 & \ell-2 \end{array}\right] = \left[\begin{array}{cc|c} 2 & 6 & 5 \\ 0 & 0 & \ell-2 \end{array}\right]$$

If $\ell - 2 \neq 0$, that is if $\ell \neq 2$, then $\operatorname{rank}(A) = 1 < 2 = \operatorname{rank}\big([\,A \mid \vec{b}\,]\big)$ so the system is inconsistent and thus has no solutions. If $\ell - 2 = 0$, that is if $\ell = 2$, then $\operatorname{rank}(A) = 1 = \operatorname{rank}\big([\,A \mid \vec{b}\,]\big)$ so the system is consistent with $2 - \operatorname{rank}(A) = 2 - 1 = 1$ parameter. Hence we have infinitely many solutions.

In summary,

$$\begin{array}{rcl} \text{Unique Solution} & : & k \neq -3 \\ \text{No Solutions} & : & k = -3 \text{ and } \ell \neq 2 \\ \text{Infinitely Many Solutions} & : & k = -3 \text{ and } \ell = 2 \end{array}$$

□

Definition 21.2. A linear system of m equations in n variables is *underdetermined* if $n > m$, this is, if it has more variables than equations.

Example 21.3. The linear system of equations

$$\begin{array}{rcrcrcrcrl} x_1 & + & x_2 & - & x_3 & + & x_4 & - & x_5 & = 1 \\ x_1 & - & x_2 & - & 3x_3 & + & 2x_4 & + & 2x_5 & = 7 \end{array}$$

is underdetermined.

Theorem 21.4. *A consistent underdetermined linear system of equations has infinitely many solutions.*

Proof. Consider a consistent underdetermined linear system of m equations in n variables with augmented matrix $[\,A\,|\,\vec{b}\,]$. Since $\operatorname{rank}(A) \leq \min\{m, n\} = m$, the system will have $n - \operatorname{rank}(A) \geq n - m > 0$ parameters and so will have infinitely many solutions. □

Definition 21.5. A linear system of m equations in n variables is *overdetermined* if $n < m$, this is, if it has more equations than variables.

Example 21.6. The linear system of equations

$$\begin{array}{rcrcr} -2x_1 & + & x_2 & = & 2 \\ x_1 & - & 3x_2 & = & 4 \\ 3x_1 & + & 2x_2 & = & 7 \end{array}$$

is overdetermined.

Note that overdetermined systems are often inconsistent. Indeed, the system in the previous example is inconsistent. To see why this is, consider for example, three lines in $\mathbb{R}^2$ (so a system of three equations in two variables like the one in the previous example). When chosen arbitrarily, it is generally unlikely that all three lines would intersect in a common point and hence we would generally expect no solutions.

Homogeneous Systems of Linear Equations

We now discuss a particular type of linear system of equations that appears quite frequently.

Definition 21.7. A *homogeneous linear equation* is a linear equation where the constant term is zero. A *system of homogeneous linear equations* is a collection of finitely many homogeneous equations.

Example 21.8. A homogeneous system of m linear equations in n variables is written as

$$\begin{array}{rcrcccrcc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

As this is still a linear system of equations, we use our usual techniques to solve such systems. However, notice that $x_1 = x_2 = \cdots = x_n = 0$ satisfies each equation in the homogeneous system, and thus $\vec{0} \in \mathbb{R}^n$ is a solution to this system, called the *trivial solution*. As every homogeneous system has a trivial solution, we see immediately that homogeneous linear systems of equations are always consistent.

Example 21.9. Solve the homogeneous linear system

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 0 \\ & & 3x_2 & - & x_3 & = & 0 \end{array}$$

Solution. We have

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 3 & -1 & 0 \end{array}\right] \underset{\frac{1}{3}R_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1/3 & 0 \end{array}\right] \underset{\longrightarrow}{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 4/3 & 0 \\ 0 & 1 & -1/3 & 0 \end{array}\right]$$

so

$$\begin{array}{rcl} x_1 & = & -\frac{4}{3}t \\ x_2 & = & \frac{1}{3}t, \quad t \in \mathbb{R} \\ x_3 & = & t \end{array} \qquad \text{or} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -4/3 \\ 1/3 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

□

We make a few remarks about this example:

- Note that taking $t = 0$ gives the trivial solution. However, as our system was underdetermined, we have infinitely many solutions. Indeed, the solution set is actually a line through the origin.

- We can simplify our solution a little bit by eliminating fractions:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -4/3 \\ 1/3 \\ 1 \end{bmatrix} = \frac{t}{3} \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix} = s \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}, \quad s \in \mathbb{R}$$

  where $s = t/3$. Hence we can let the parameter "absorb" the factor of $1/3$. This is not necessary, but is useful if one wishes to eliminate fractions.

- When working with homogeneous systems of linear equations, notice that the augmented matrix $[\, A \,|\, \vec{0} \,]$ will always have the last column containing all zero entries. Thus, it is common to row reduce only the coefficient matrix.

Given a non-homogeneous linear system of equations with augmented matrix $[\, A \,|\, \vec{b} \,]$ (so $\vec{b} \neq \vec{0}$), the homogeneous system with augmented matrix $[\, A \,|\, \vec{0} \,]$ is called the *associated homogeneous system*. The solution to the associated homogeneous system tells us a lot about the solution of the original non-homogeneous system.

Example 21.10. If we solve the system

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 1 \\ & & 3x_2 & - & x_3 & = & 3 \end{array}$$

we obtain

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 3 & -1 & 3 \end{array}\right] \underset{\frac{1}{3}R_2}{\longrightarrow} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 1 & -1/3 & 1 \end{array}\right] \overset{R_1-R_2}{\longrightarrow} \left[\begin{array}{rrr|r} 1 & 0 & 4/3 & 0 \\ 0 & 1 & -1/3 & 1 \end{array}\right]$$

so

$$\begin{array}{rcl} x_1 & = & -\frac{4}{3}t \\ x_2 & = & 1+\frac{1}{3}t, \qquad t \in \mathbb{R} \\ x_3 & = & t \end{array} \qquad \text{or} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4/3 \\ 1/3 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Note that the solution to the associated homogeneous system (from Example 21.9) is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -4/3 \\ 1/3 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

so we view the homogeneous solution from Example 21.9 as a line, say L_0, through the origin, and the solution from Example 21.10 as a line, say L_1, through $P(0,1,0)$ parallel to L_0. We refer to $\left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right]$ as a *particular solution* to the system in Example 21.10 and note that in general, the solution to a consistent non-homogeneous system of linear equations is a particular solution plus the solution to the associated homogeneous system of linear equations.

$$\overbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\substack{\text{particular}\\ \text{solution}}} + t \underbrace{\begin{bmatrix} -4/3 \\ 1/3 \\ 1 \end{bmatrix}}_{\substack{\text{associated}\\ \text{homogeneous}\\ \text{solution}}}, \quad t \in \mathbb{R}}^{\text{solution to the system of equations}} \qquad \overbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -4/3 \\ 1/3 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}}^{\substack{\text{solution to the}\\ \text{associated homogeneous system of equations}}}.$$

Example 21.11. Consider the system of linear equations

$$\begin{array}{rcrcrcrcr} x_1 & + & 6x_2 & & & - & x_4 & = & -1 \\ & & & & x_3 & + & 2x_4 & = & 7 \end{array}$$

We know from Example 19.4 that the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 7 \\ 0 \end{bmatrix} + s \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R},$$

so the solution to the associated homogeneous system

$$\begin{array}{rcrcrcrcr} x_1 & + & 6x_2 & & & - & x_4 & = & 0 \\ & & & & x_3 & + & 2x_4 & = & 0 \end{array}$$

is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

which we recognize as a plane through the origin in $\mathbb{R}^4$ since the two vectors appearing in the solution are nonzero and nonparallel.

From Examples 21.9 and 21.11 we saw that our solutions sets were lines and planes through the origin which we recognize as subspaces. The following theorem shows that the solution set to any homogeneous system in n variables will indeed be a subspace of $\mathbb{R}^n$.

Theorem 21.12. *Let $\mathbb{S}$ be the solution set to a homogeneous system of m linear equations in n variables. Then $\mathbb{S}$ is a subspace of $\mathbb{R}^n$.*

Proof. Since the system has n variables, $\mathbb{S} \subseteq \mathbb{R}^n$ and since the system is homogeneous, $\vec{0} \in \mathbb{S}$ so $\mathbb{S}$ is nonempty. Now let

$$\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad \text{and} \quad \vec{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

be vectors in $\mathbb{S}$. To show that $\mathbb{S}$ is closed under vector addition and scalar multiplication, it is enough to consider one arbitrary equation of the system:

$$a_1 x_1 + \cdots + a_n x_n = 0.$$

Then $\vec{y}, \vec{z} \in \mathbb{S}$ imply that

$$a_1 y_1 + \cdots + a_n y_n = 0 = a_1 z_1 + \cdots + a_n z_n.$$

It follows that

$$a_1(y_1+z_1)+\cdots+a_n(y_n+z_n)=a_1y_1+\cdots+a_ny_n+a_1z_1+\cdots+a_nz_n=0+0=0$$

so $\vec{y}+\vec{z}$ satisfies any equation of the system and thus $\vec{y}+\vec{z}\in\mathbb{S}$. For $c\in\mathbb{R}$,

$$a_1(cy_1)+\cdots+a_n(cy_n)=c(a_1y_1+\cdots+a_ny_n)=c(0)=0$$

so $c\vec{y}\in\mathbb{S}$. Hence $\mathbb{S}$ is a subspace of $\mathbb{R}^n$. □

Note that we call the solution set of a homogeneous system the *solution space* of the system.

Example 21.13. Solve the homogeneous system of linear equations

$$\begin{array}{rcrcrcrcl} 4x_1 & - & 2x_2 & + & 3x_3 & + & 5x_4 & = & 0 \\ 8x_1 & - & 4x_2 & + & 6x_3 & + & 11x_4 & = & 0 \\ -4x_1 & + & 2x_2 & - & 3x_3 & - & 7x_4 & = & 0 \end{array}$$

and find a basis for the solution space $\mathbb{S}$. Describe $\mathbb{S}$ geometrically.

Solution. As we have a homogeneous system, we carry the coefficient matrix to RREF.

$$\begin{bmatrix} 4 & -2 & 3 & 5 \\ 8 & -4 & 6 & 11 \\ -4 & 2 & -3 & -7 \end{bmatrix} \underset{\substack{R_2-2R_1 \\ R_3+R_1}}{\longrightarrow} \begin{bmatrix} 4 & -2 & 3 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \underset{R_3+2R_2}{\overset{R_1-5R_2}{\longrightarrow}} \begin{bmatrix} 4 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{\frac{1}{4}R_1}{\longrightarrow}$$
$$\begin{bmatrix} 1 & -1/2 & 3/4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so

$$\begin{array}{rcl} x_1 & = & \frac{1}{2}s-\frac{3}{4}t \\ x_2 & = & s \\ x_3 & = & t \\ x_4 & = & 0 \end{array}, \quad s,t\in\mathbb{R} \qquad \text{or} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3/4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad s,t\in\mathbb{R}.$$

Taking

$$B=\left\{\begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/4 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\},$$

we can express the solution set $\mathbb{S}$ of our homogeneous system of linear equations as $\mathbb{S}=\operatorname{Span} B$. As B contains two vectors that are not scalar multiples of one another, we have that B is a basis for $\mathbb{S}$. We see that $\mathbb{S}$ is a plane through the origin in $\mathbb{R}^4$. □

Lecture 22

Consider the homogeneous system of linear equations

$$\begin{array}{ccccccccccc} x_1 & + & x_2 & + & x_3 & & & + & 4x_5 & = & 0 \\ & & & & & & x_4 & + & 2x_5 & = & 0 \end{array}$$

The coefficient matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

is already in reduced row echelon form[36] and our solution is

$$\begin{array}{rcl} x_1 & = & -t_1 - t_2 - 4t_3 \\ x_2 & = & t_1 \\ x_3 & = & t_2 \\ x_4 & = & -2t_3 \\ x_5 & = & t_3 \end{array} \qquad \text{or} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -4 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

with $t_1, t_2, t_3 \in \mathbb{R}$ so

$$B = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

is a spanning set for the solution space $\mathbb{S}$ of the system. We check B for linear independence. Note however that the variables x_2, x_3 and x_5 are free variables. If we consider the second, third and fifth entries in vectors of our spanning set

$$B = \left\{ \begin{bmatrix} -1 \\ \color{red}{1} \\ \color{blue}{0} \\ 0 \\ \color{brown}{0} \end{bmatrix}, \begin{bmatrix} -1 \\ \color{red}{0} \\ \color{blue}{1} \\ 0 \\ \color{brown}{0} \end{bmatrix}, \begin{bmatrix} -4 \\ \color{red}{0} \\ \color{blue}{0} \\ -2 \\ \color{brown}{1} \end{bmatrix} \right\}$$

we see that each vector has a 1 where the other two vectors have zeros in that same position. Thus no vector in B is in the span of the others, and so B is linearly independent by Theorem 14.5. Hence B is a basis for the solution space $\mathbb{S}$.

[36] Remember that for homogeneous systems of linear equations, we normally row reduce just the coefficient matrix.

Theorem 22.1. *Let* $[\,A\,|\,\vec{b}\,]$ *be the augmented matrix for a consistent system of* m *linear equations in* n *variables. If* $\operatorname{rank}(A) = k < n$*, then the general solution of the system is of the form*

$$\vec{x} = \vec{d} + t_1\vec{v}_1 + \cdots + t_{n-k}\vec{v}_{n-k}$$

where $\vec{d} \in \mathbb{R}^n$*,* $t_1, \ldots, t_{n-k} \in \mathbb{R}$ *and the set* $\{\vec{v}_1, \ldots, \vec{v}_{n-k}\} \subseteq \mathbb{R}^n$ *is linearly independent. In particular, the solution set is an* $(n-k)-$*flat in* $\mathbb{R}^n$*.*

Note that if $\operatorname{rank}(A) = n$ in the above theorem, then there are $n - n = 0$ parameters and so our solution $\vec{x} = \vec{d}$ is unique.

When solving a homogeneous system of linear equations, we see that the spanning set for the solution space we find by solving the system is linearly independent. However, given an arbitrary spanning set B for a subspace of $\mathbb{R}^n$, we cannot assume that B is linearly independent, and so we must still check. We now show a faster way to do so.
Consider

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 6 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{v}_4 = \begin{bmatrix} 5 \\ 7 \\ 12 \\ 17 \end{bmatrix}$$

and let $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ and $\mathbb{S} = \operatorname{Span} B$. We wish to find a basis B' for $\mathbb{S}$ with $B' \subseteq B$. That is, find a linearly independent subset B' of B with $\operatorname{Span} B' = \mathbb{S}$. For $c_1, c_2, c_3, c_4 \in \mathbb{R}$, considering

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$$

gives a homogeneous system whose coefficient matrix we carry to RREF:

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 1 & 2 & 2 & 7 \\ 2 & 4 & 3 & 12 \\ 3 & 6 & 4 & 17 \end{bmatrix} \begin{matrix} \longrightarrow \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - 3R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \longrightarrow \\ R_3 - R_2 \\ R_4 - R_2 \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that c_2 and c_4 are free variables so we obtain nontrivial solutions to the system and hence B is linearly dependent. Our work with bases thus far has shown us that since we can find solutions with $c_2 \neq 0$ and $c_4 \neq 0$, we can remove one of $\vec{v}_2$ or $\vec{v}_4$ from B and then test the resulting smaller set for linear independence. We show here that we can simply remove both $\vec{v}_2$ and $\vec{v}_4$ and arrive at $B' = \{\vec{v}_1, \vec{v}_3\}$ as our basis for $\mathbb{S}$ immediately.

To begin, note that c_1 and c_3 were leading variables in the above system. Using our work above, we see that by considering the homogeneous system

$$c_1\vec{v}_1 + c_3\vec{v}_3 = \vec{0}$$

we obtain

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which has only the trivial solution so $\{\vec{v}_1, \vec{v}_3\}$ is linearly indepedent. If we try to write $\vec{v}_4$ as a linear combination of $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$, we obtain the system with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 1 & 2 & 2 & 7 \\ 2 & 4 & 3 & 12 \\ 3 & 6 & 4 & 17 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent (with infinitely many solutions), so $\vec{v}_4 \in \operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and so by Theorem 13.8, $\operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ so we "discard" $\vec{v}_4$. Now, if we try to express $\vec{v}_2$ as a linear combination of $\vec{v}_1$, we obtain the system with augmented matrix

$$\left[\begin{array}{c|c} 1 & 2 \\ 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{c|c} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right]$$

which is also consistent (with a unique solution) so $\vec{v}_2 \in \operatorname{Span}\{\vec{v}_1\} \subseteq \operatorname{Span}\{\vec{v}_1, \vec{v}_3\}$ and we have that $\operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{Span}\{\vec{v}_1, \vec{v}_3\}$ by Theorem 13.8. We will thus "discard" $\vec{v}_2$. In summary, we've shown

$$\mathbb{S} = \operatorname{Span} B = \operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{Span}\{\vec{v}_1, \vec{v}_3\}$$

with $\{\vec{v}_1, \vec{v}_3\}$ linearly independent. Hence $B' = \{\vec{v}_1, \vec{v}_3\}$ is a basis for $\mathbb{S}$.

Thus, we see that given a spanning set $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ for a subspace $\mathbb{S}$ of $\mathbb{R}^n$, to find a basis B' for $\mathbb{S}$ with $B' \subseteq B$, we construct the matrix $[\,\vec{v}_1 \cdots \vec{v}_k\,]$ which we carry to (reduced) row echelon form. For $i = 1, \ldots, k$, take $\vec{v}_i \in B'$ if and only if the ith column of any REF of our matrix has a leading entry. We also see that for $\vec{v}_j \notin B'$, $\vec{v}_j$ can be expressed as a linear combination of the vectors in $\{\vec{v}_1, \ldots, \vec{v}_{j-1}\} \cap B'$.

Example 22.2. Let

$$B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix} \right\}.$$

Find a basis B' for $\operatorname{Span} B$ with $B' \subseteq B$.

Solution. We have

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ -1 & 2 & 5 & 6 \\ 1 & -3 & -7 & -9 \end{bmatrix} \underset{\substack{R_2+R_1 \\ R_3-R_1}}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 6 & 9 \\ 0 & -4 & -8 & -12 \end{bmatrix} \underset{R_3+\frac{4}{3}R_2}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As only the first two columns of an REF of our matrix contain leading entries, the first two vectors in B comprise B', that is

$$B' = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right\}$$

is a basis for $\operatorname{Span} B$. □

Note that if we had continued to row reduce to RREF, we would have found

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ -1 & 2 & 5 & 6 \\ 1 & -3 & -7 & -9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (\star)$$

Note that the third and fourth columns of the RREF do not contain leading ones. We see that those vectors in B not taken in B' satisfy

$$\begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \text{ since } \left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 2 & 5 \\ 1 & -3 & -7 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \left(\begin{array}{l} \text{omit 4th} \\ \text{columns from} \\ \text{matrices in } (\star) \end{array} \right)$$

$$\begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \text{ since } \left[\begin{array}{cc|c} 1 & 1 & 3 \\ -1 & 2 & 6 \\ 1 & -3 & -9 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \left(\begin{array}{l} \text{omit 3rd} \\ \text{columns from} \\ \text{matrices in } (\star) \end{array} \right)$$

Dimension

Let $\mathbb{S}$ be a subspace of $\mathbb{R}^n$ and $B = \{\vec{v}_1, \vec{v}_2\}$ be a basis for $\mathbb{S}$. If $C = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is a set of vectors in $\mathbb{S}$, then C must be linearly dependent. To see this, note that since B is a basis for $\mathbb{S}$, Theorem 15.6 gives that there are unique $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$ so that

$$\vec{w}_1 = a_1\vec{v}_1 + a_2\vec{v}_2, \quad \vec{w}_2 = b_1\vec{v}_1 + b_2\vec{v}_2 \quad \text{and} \quad \vec{w}_3 = c_1\vec{v}_1 + c_2\vec{v}_2.$$

Now for $t_1, t_2, t_3 \in \mathbb{R}$, consider

$$\vec{0} = t_1\vec{w}_1 + t_2\vec{w}_2 + t_3\vec{w}_3$$

$$
\begin{aligned}
&= t_1(a_1\vec{v}_1 + a_2\vec{v}_2) + t_2(b_1\vec{v}_1 + b_2\vec{v}_2) + t_3(c_1\vec{v}_1 + c_2\vec{v}_2) \\
&= (a_1t_1 + b_1t_2 + c_1t_3)\vec{v}_1 + (a_2t_1 + b_2t_2 + c_2t_3)\vec{v}_2
\end{aligned}
$$

Since $B = \{\vec{v}_1, \vec{v}_2\}$ is linearly independent we have,

$$
\begin{array}{ccccccc}
a_1t_1 & + & b_1t_2 & + & c_1t_3 & = & 0 \\
a_2t_1 & + & b_2t_2 & + & c_2t_3 & = & 0
\end{array}
$$

This is an underdetermined homogeneous system, so it is consistent with nontrivial solutions and it follows that $C = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is linearly dependent.

The above generalizes as follows.

Theorem 22.3. *Let $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for a subspace $\mathbb{S}$ of $\mathbb{R}^n$. If $C = \{\vec{w}_1, \dots, \vec{w}_\ell\}$ is a set in $\mathbb{S}$ with $\ell > k$, then C is linearly dependent.*

It follows from the statement of the previous theorem that if C is linearly independent, then $\ell \leq k$. We now state the following important result:

Theorem 22.4. *If $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ and $C = \{\vec{w}_1, \dots, \vec{w}_\ell\}$ are both bases for a subspace $\mathbb{S}$ of $\mathbb{R}^n$, then $k = \ell$.*

Proof. Since B is a basis for $\mathbb{S}$ and C is linearly independent, we have that $\ell \leq k$. Since C is a basis for $\mathbb{S}$ and B is linearly independent, $k \leq \ell$. Hence $k = \ell$. □

Hence, given a nontrivial subspace $\mathbb{S}$ of $\mathbb{R}^n$, there are many bases for $\mathbb{S}$, but they will all contain the same number of vectors. This motivates the following definition.

Definition 22.5. If $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis for a subspace $\mathbb{S}$ of $\mathbb{R}^n$, then we say the *dimension* of $\mathbb{S}$ is k, and we write $\dim(\mathbb{S}) = k$. If $\mathbb{S} = \{\vec{0}\}$, then $\dim(\mathbb{S}) = 0$ since $\emptyset$ is a basis for $\mathbb{S}$.

Example 22.6. Since the standard basis for $\mathbb{R}^n$ is $\{\vec{e}_1, \dots, \vec{e}_n\}$, we see that $\dim(\mathbb{R}^n) = n$.

Example 22.7. We saw in Example 16.8 that the subspace

$$
\mathbb{S} = \left\{ \begin{bmatrix} a-b \\ b-c \\ c-a \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}
$$

of $\mathbb{R}^3$ had basis

$$
B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}
$$

so $\dim(\mathbb{S}) = 2$.

Theorem 22.8. *If $\mathbb{S}$ is a $k-$dimensional subspace of $\mathbb{R}^n$ with $k > 0$, then*

(1) *A set of more than k vectors in $\mathbb{S}$ is linearly dependent,*

(2) *A set of fewer than k vectors in $\mathbb{S}$ cannot span $\mathbb{S}$,*

(3) *A set of exactly k vectors in $\mathbb{S}$ spans $\mathbb{S}$ if and only if it is linearly independent.*

Example 22.9. Let $\mathbb{S}$ be a subspace of $\mathbb{R}^3$ with $\dim(\mathbb{S}) = 2$. Suppose that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

belong to $\mathbb{S}$. Since $\vec{v}_1$ and $\vec{v}_2$ are nonzero and nonparallel, we have that $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set of two vectors in $\mathbb{S}$. Since $\dim(\mathbb{S}) = 2$, we have that $\mathbb{S} = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$ by Theorem 22.8(3). Thus $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbb{S}$.

Note that we must know $\dim(\mathbb{S})$ *before* we use Theorem 22.8. In the previous example, we could not have used the linear independence of $\{\vec{v}_1, \vec{v}_2\}$ to conclude that $\mathbb{S} = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$ if we weren't told the dimension of $\mathbb{S}$.

Lecture 23

We now begin to look at some application of systems of linear equations.

Application: Chemical Reactions

A very simple chemical reaction often learned in high school is the combination of hydrogen molecules (H_2) and oxygen molecules (O_2) to produce water (H_2O). Symbolically, we write

$$\mathrm{H_2 + O_2 \longrightarrow H_2O}$$

The process by which molecules combine to form new molecules is called a *chemical reaction*. Note that each hydrogen molecule is composed of two hydrogen atoms, each oxygen molecule is composed of two oxygen atoms, and that each water molecule is composed of two hydrogen atoms and one oxygen atom. Our goal is to *balance* this chemical reaction, that is, compute how many hydrogen molecules and how many oxygen molecules are needed so that there are the same number of atoms of each type both before and after the chemical reaction takes place. By inspection, we find that

$$\mathrm{2H_2 + O_2 \longrightarrow 2H_2O}$$

That is, two hydrogen molecules and one oxygen molecule combine to create two water molecules. Before this chemical reaction takes place, there are four hydrogen atoms and two oxygen atoms. After the reaction, there are again four hydrogen atoms and two oxygen atoms. Thus we have balanced the chemical reaction.

Balancing chemical reactions by inspection becomes increasingly difficult as more complex molecules are introduced. For example, the chemical reaction *photosynthesis* is a process where plants combine carbon dioxide (CO_2) and water (H_2O) to produce glucose ($C_6H_{12}O_6$) and oxygen (O_2):

$$\mathrm{CO_2 + H_2O \longrightarrow C_6H_{12}O_6 + O_2}$$

Although this could be solved by inspection, we look at another method. Let x_1 denote the number of CO_2 molecules, x_2 the number of H_2O molecules, x_3 the number of $C_6H_{12}O_6$ molecules and x_4 the number of O_2 molecules. Then we have

$$x_1\mathrm{CO_2} + x_2\mathrm{H_2O} \longrightarrow x_3\mathrm{C_6H_{12}O_6} + x_4\mathrm{O_2}$$

Equating the number of atoms of each type before and after the reaction gives the equations

$$\begin{aligned}
\mathrm{C}: \quad & x_1 &= \;& 6x_3 \\
\mathrm{O}: \quad & 2x_1 + x_2 &= \;& 6x_3 + 2x_4 \\
\mathrm{H}: \quad & 2x_2 &= \;& 12x_3
\end{aligned}$$

Moving all variables to the left in each equation gives the homogeneous system

$$\begin{array}{rcrcrcrcl} x_1 & & & - & 6x_3 & & & = & 0 \\ 2x_1 & + & x_2 & - & 6x_3 & - & 2x_4 & = & 0 \\ & & 2x_2 & - & 12x_3 & & & = & 0 \end{array}$$

Row reducing the augmented matrix of this system to RREF gives

$$\left[\begin{array}{rrrr|r} 1 & 0 & -6 & 0 & 0 \\ 2 & 1 & -6 & -2 & 0 \\ 0 & 2 & -12 & 0 & 0 \end{array}\right] \underset{\frac{1}{2}R_3}{\overset{\longrightarrow}{R_2-2R_1}} \left[\begin{array}{rrrr|r} 1 & 0 & -6 & 0 & 0 \\ 0 & 1 & 6 & -2 & 0 \\ 0 & 1 & -6 & 0 & 0 \end{array}\right] \underset{R_3-R_2}{\longrightarrow} \left[\begin{array}{rrrr|r} 1 & 0 & -6 & 0 & 0 \\ 0 & 1 & 6 & -2 & 0 \\ 0 & 0 & -12 & 2 & 0 \end{array}\right] \underset{-\frac{1}{2}R_3}{\longrightarrow}$$

$$\left[\begin{array}{rrrr|r} 1 & 0 & -6 & 0 & 0 \\ 0 & 1 & 6 & -2 & 0 \\ 0 & 0 & 6 & -1 & 0 \end{array}\right] \begin{array}{c} R_1+R_3 \\ R_2-R_3 \\ \longrightarrow \end{array} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 6 & -1 & 0 \end{array}\right] \underset{\frac{1}{6}R_3}{\longrightarrow} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1/6 & 0 \end{array}\right]$$

We see that for $t \in \mathbb{R}$,

$$x_1 = t, \quad x_2 = t, \quad x_3 = t/6 \quad \text{and} \quad x_4 = t$$

There are infinitely many solutions to the homogeneous system. However, since we cannot have a fractional number of molecules, we require that x_1, x_2, x_3 and x_4 be nonnegative integers. This implies that t should be an integer multiple of 6. Moreover, we wish to have the simplest (or smallest) solution, so we will take $t = 6$. This gives $x_1 = x_2 = x_4 = 6$ and $x_3 = 1$. Thus,

$$6CO_2 + 6H_2O \longrightarrow C_6H_{12}O_6 + 6O_2$$

balances the chemical reaction.

Example 23.1. The *fermentation of sugar* is a chemical reaction given by the following equation:

$$C_6H_{12}O_6 \longrightarrow CO_2 + C_2H_5OH$$

where $C_6H_{12}O_6$ is glucose, CO_2 is carbon dioxide and C_2H_5OH is ethanol[37]. Balance this chemical reaction.

Solution. Let x_1 denote the number of $C_6H_{12}O_6$ molecules, x_2 the number of CO_2 molecules and x_3 the number of C_2H_5OH molecules. We obtain

$$x_1C_6H_{12}O_6 \longrightarrow x_2CO_2 + x_3C_2H_5OH$$

Equating the number of atoms of each type before and after the reaction gives the equations

$$\begin{array}{rrcl} \text{C}: & 6x_1 & = & x_2 + 2x_3 \\ \text{O}: & 6x_1 & = & 2x_2 + x_3 \\ \text{H}: & 12x_1 & = & 6x_3 \end{array}$$

[37]Ethanol is also denoted by C_2H_6O and CH_3CH_2OH

which leads to the homogeneous system of equations

$$\begin{array}{rcrcrcl} 6x_1 & - & x_2 & - & 2x_3 & = & 0 \\ 6x_1 & - & 2x_2 & - & x_3 & = & 0 \\ 12x_1 & & & - & 6x_3 & = & 0 \end{array}$$

Carrying the augmented matrix of this system to RREF gives

$$\left[\begin{array}{rrr|r} 6 & -1 & -2 & 0 \\ 6 & -2 & -1 & 0 \\ 12 & 0 & -6 & 0 \end{array}\right] \begin{array}{c} \longrightarrow \\ R_2-R_1 \\ R_3-2R_1 \end{array} \left[\begin{array}{rrr|r} 6 & -1 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right] \begin{array}{c} R_1-R_2 \\ \longrightarrow \\ R_3+2R_2 \end{array} \left[\begin{array}{rrr|r} 6 & 0 & -3 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \frac{1}{6}R_1 \\ -R_2 \\ \longrightarrow \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, for $t \in \mathbb{R}$,

$$x_1 = t/2, \quad x_2 = t \quad \text{and} \quad x_3 = t$$

Taking $t = 2$ gives the smallest nonnegative integer solution, and we conclude that

$$C_6H_{12}O_6 \longrightarrow 2CO_2 + 2C_2H_5OH$$

□

Application: Linear Models

Example 23.2. An industrial city has four heavy industries (denoted by A_1, A_2, A_3, A_4) each of which burns coal to manufacture its products. By law, no industry can burn more than 45 units of coal per day. Each industry produces the pollutants Pb (lead), SO_2 (sulfur dioxide), and NO_2 (nitrogen dioxide) at (different) daily rates per unit of coal burned and these are released into the atmosphere. The rates are shown in the following table.

Industry	A_1	A_2	A_3	A_4
Pb	1	0	1	7
SO_2	2	1	2	9
NO_2	0	2	2	0

The CAAG (Clean Air Action Group) has just leaked a government report that claims that on one day last year, 250 units of Pb, 550 units of SO_2 and 400 units of NO_2 were measured in the atmosphere. An inspector reported that A_3 did not break the law on that day. Which industry (or industries) broke the law on that day?

Solution. Let a_i denote the number of units of coal burned by Industry A_i, for $i = 1, 2, 3, 4$. Using the above table, we account for each of the pollutants on that day.

$$\begin{array}{rcccccccl} \text{Pb}: & a_1 & + & & & a_3 & + & 7a_4 & = & 250 \\ \text{SO}_2: & 2a_1 & + & a_2 & + & 2a_3 & + & 9a_4 & = & 550 \\ \text{NO}_2: & & & 2a_2 & + & 2a_3 & & & = & 400 \end{array}$$

Carrying the augmented matrix of the above system to RREF, we have

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 7 & 250 \\ 2 & 1 & 2 & 9 & 550 \\ 0 & 2 & 2 & 0 & 400 \end{array}\right] \underset{R_2-2R_1}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 7 & 250 \\ 0 & 1 & 0 & -5 & 50 \\ 0 & 2 & 2 & 0 & 400 \end{array}\right] \underset{R_3-2R_2}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 7 & 250 \\ 0 & 1 & 0 & -5 & 50 \\ 0 & 0 & 2 & 10 & 300 \end{array}\right] \underset{\frac{1}{2}R_3}{\longrightarrow}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 7 & 250 \\ 0 & 1 & 0 & -5 & 50 \\ 0 & 0 & 1 & 5 & 150 \end{array}\right] \overset{R_1-R_3}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 100 \\ 0 & 1 & 0 & -5 & 50 \\ 0 & 0 & 1 & 5 & 150 \end{array}\right]$$

From this, we find that

$$a_1 = 100 - 2t, \quad a_2 = 50 + 5t, \quad a_3 = 150 - 5t, \quad a_4 = t$$

where $t \in \mathbb{R}$. Now we look for conditions on t. We know A_3 did not break that law, so $0 \leq a_3 \leq 45$, that is,

$$\begin{array}{rcccl} 0 & \leq & 150 - 5t & \leq & 45 \\ -150 & \leq & -5t & \leq & -105 \\ 30 & \geq & t & \geq & 21 \end{array}$$

It immediately follows that A_4 didn't break that law as $a_4 = t$. Looking at A_2, we have

$$\begin{array}{rcccl} 21 & \leq & t & \leq & 30 \\ 105 & \leq & 5t & \leq & 150 \\ 155 & \leq & 50 + 5t & \leq & 200 \\ 155 & \leq & a_2 & \leq & 200 \end{array}$$

so A_2 broke the law. Finally, for A_1, we find

$$\begin{array}{rcccl} 21 & \leq & t & \leq & 30 \\ -42 & \geq & -2t & \geq & -60 \\ 58 & \geq & 100 - 2t & \geq & 40 \\ 58 & \geq & a_1 & \geq & 40 \end{array}$$

so it is *possible* that A_1 broke the law, but we cannot be sure without more information. □

Example 23.3. An engineering company has three divisions (Design, Production, Testing) with a combined annual budget of \$1.5 million. Production has an annual budget equal to the combined annual budgets of Design and Testing. Testing requires a budget of at least \$80 000. What is the Production budget and the maximum possible budget for the Design division?

Solution. Let x_1 denote the annual Design budget, x_2 the annual Production budget, and x_3 the annual Testing budget. It follows that $x_1 + x_2 + x_3 = 1\,500\,000$. Since the annual Production budget is equal the the combined Design and Testing budgets, we have $x_2 = x_1 + x_3$. This gives the system of equations

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 1\,500\,000 \\ x_1 & - & x_2 & + & x_3 & = & 0 \end{array}$$

Row reducing the above system gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\,500\,000 \\ 1 & -1 & 1 & 0 \end{array}\right] \underset{R_2-R_1}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\,500\,000 \\ 0 & -2 & 0 & -1\,500\,000 \end{array}\right] \underset{-\frac{1}{2}R_2}{\longrightarrow}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\,500\,000 \\ 0 & 1 & 0 & 750\,000 \end{array}\right] \overset{R_1-R_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 750\,000 \\ 0 & 1 & 0 & 750\,000 \end{array}\right]$$

This gives

$$x_1 = 750\,000 - t, \qquad x_2 = 750\,000, \qquad x_3 = t$$

where $t \in \mathbb{R}$. We know that the Testing budget requires at least \$80 000 and can receive no more than \$750 000 (since Testing shares a budget of \$750 000 with Design). Thus $80\,000 \leq t \leq 750\,000$. It follows that

$$\begin{array}{ccccc} -750\,000 & \leq & -t & \leq & -80\,000 \\ 0 & \leq & 750\,000 - t & \leq & 670\,000 \\ 0 & \leq & x_1 & \leq & 670\,000 \end{array}$$

Hence the Production budget is \$750 000 and the maximum Design budget is \$670 000. □

Lecture 24

Application: Network Flow

A *network* consists of a system of *junctions* or *nodes* that are connected by *directed line segments*. These networks are used to model real world problems such as traffic flow, fluid flow, or any such system where a flow is observed. We observe here the central rule that must be obeyed by these systems.

Junction Rule: At each of the junctions (or nodes) in the network, the flow into that junction must equal the flow out of that junction.

Our goal is to achieve a network such that every junction obeys the Junction Rule. We say that such a system is in a *steady state* or *equilibrium*.

Figure 47 below gives an example of a network with four nodes, A, B, C and D, and eight directed line segments. We wish to compute all possible values of f_1, f_2, f_3 and f_4 so that the system is in equilibrium.

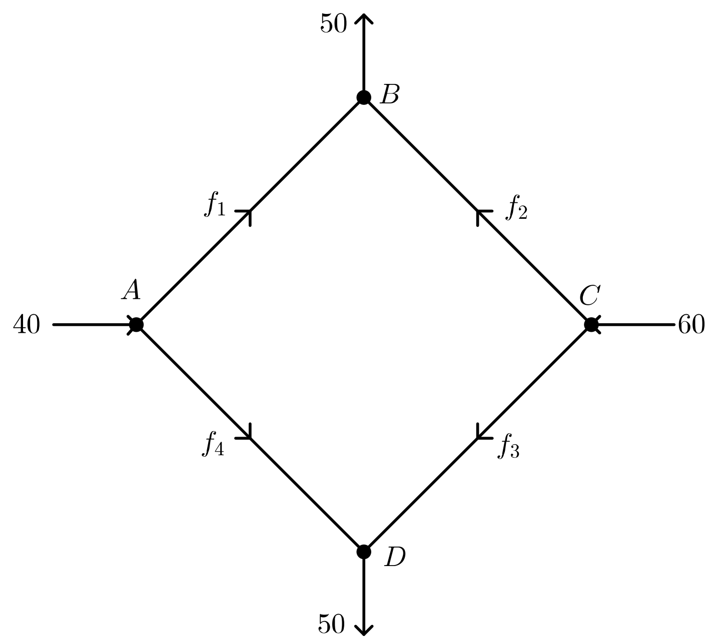


Figure 47: A simple network

Using the Junction Rule at each node, we construct the following table:

	Flow In		Flow Out
$A:$	40	$=$	$f_1 + f_4$
$B:$	$f_1 + f_2$	$=$	50
$C:$	60	$=$	$f_2 + f_3$
$D:$	$f_3 + f_4$	$=$	50

Rearranging each of the above four linear equations leads to the following system:

$$\begin{array}{rcrcrcrcr} f_1 & & & & & + & f_4 & = & 40 \\ f_1 & + & f_2 & & & & & = & 50 \\ & & f_2 & + & f_3 & & & = & 60 \\ & & & & f_3 & + & f_4 & = & 50 \end{array}$$

Row reducing the augmented matrix to RREF, we have

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 40 \\ 1 & 1 & 0 & 0 & 50 \\ 0 & 1 & 1 & 0 & 60 \\ 0 & 0 & 1 & 1 & 50 \end{array}\right] \xrightarrow[R_2-R_1]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 40 \\ 0 & 1 & 0 & -1 & 10 \\ 0 & 1 & 1 & 0 & 60 \\ 0 & 0 & 1 & 1 & 50 \end{array}\right] \xrightarrow[R_3-R_2]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 40 \\ 0 & 1 & 0 & -1 & 10 \\ 0 & 0 & 1 & 1 & 50 \\ 0 & 0 & 1 & 1 & 50 \end{array}\right] \xrightarrow[R_4-R_3]{}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 40 \\ 0 & 1 & 0 & -1 & 10 \\ 0 & 0 & 1 & 1 & 50 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We find that

$$f_1 = 40 - t, \quad f_2 = 10 + t, \quad f_3 = 50 - t \quad \text{and} \quad f_4 = t$$

where $t \in \mathbb{R}$. We see that there are infinitely many values for f_1, f_2, f_3 and f_4 so that the system is in equilibrium. Note that a negative solution for one of the variables means that the flow is in the opposite direction than the one indicated in the diagram. Depending on what the network is representing, we may require that each of f_1, f_2, f_3 and f_4 be nonnegative. In this case,

$$\begin{array}{rcrcr} f_1 \geq 0 & \Longrightarrow & 40 - t \geq 0 & \Longrightarrow & t \leq 40 \\ f_2 \geq 0 & \Longrightarrow & 10 + t \geq 0 & \Longrightarrow & t \geq -10 \\ f_3 \geq 0 & \Longrightarrow & 50 - t \geq 0 & \Longrightarrow & t \leq 50 \\ f_4 \geq 0 & \Longrightarrow & t \geq 0 & & \end{array}$$

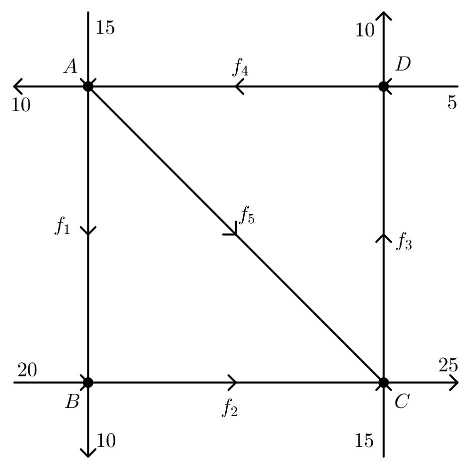
Here, we see that $0 \leq t \leq 40$. They may be more constraints on f_1, f_2, f_3 and f_4. For example, if the flows in the above network represent the number of automobiles moving between the junctions, then we further require f_1, f_2, f_3 and f_4 to be integers. In our example, this would make $t = 0, 1, 2, \ldots 40$, giving us 41 possible solutions.

When using linear algebra to model real world problems, we must be able to interpret our solutions in terms of the problem it is modelling. This includes incorporating any real world restrictions imposed by the system we are modelling.

Example 24.1. Consider four train stations labelled A, B, C and D. In the figure below, the directed line segments represent train tracks to and from stations, and the numbers represent the number of trains travelling on that track per day. Assume the tracks are one-way, so trains may not travel in the other direction.

a) Find all values of $f_1, \ldots, f_5$ so that the system is in equilibrium.

b) Suppose the tracks from A to C and from D to A are closed due to maintenance. Is it still possible for the system to be in equilibrium?



Solution.

a) We construct a table:

	Flow In		Flow Out
$A:$	$15 + f_4$	$=$	$10 + f_1 + f_5$
$B:$	$20 + f_1$	$=$	$10 + f_2$
$C:$	$15 + f_2 + f_5$	$=$	$25 + f_3$
$D:$	$5 + f_3$	$=$	$10 + f_4$

Rearranging gives the linear system of equations

$$\begin{array}{ccccccccccc} f_1 & & & & & - & f_4 & + & f_5 & = & 5 \\ f_1 & - & f_2 & & & & & & & = & -10 \\ & & f_2 & - & f_3 & & & + & f_5 & = & 10 \\ & & & & f_3 & - & f_4 & & & = & 5 \end{array}$$

which we carry to RREF

$$\left[\begin{array}{rrrrr|r} 1 & 0 & 0 & -1 & 1 & 5 \\ 1 & -1 & 0 & 0 & 0 & -10 \\ 0 & 1 & -1 & 0 & 1 & 10 \\ 0 & 0 & 1 & -1 & 0 & 5 \end{array}\right] \begin{array}{c} \longrightarrow \\ -R_2 \\ -R_3 \\ -R_4 \end{array} \left[\begin{array}{rrrrr|r} 1 & 0 & 0 & -1 & 1 & 5 \\ -1 & 1 & 0 & 0 & 0 & 10 \\ 0 & -1 & 1 & 0 & -1 & -10 \\ 0 & 0 & -1 & 1 & 0 & -5 \end{array}\right] \begin{array}{c} \longrightarrow \\ R_2+R_1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{rrrrr|r} 1 & 0 & 0 & -1 & 1 & 5 \\ 0 & 1 & 0 & -1 & 1 & 15 \\ 0 & -1 & 1 & 0 & -1 & -10 \\ 0 & 0 & -1 & 1 & 0 & -5 \end{array}\right] \begin{array}{c} \longrightarrow \\ \\ R_3+R_2 \\ \\ \end{array} \left[\begin{array}{rrrrr|r} 1 & 0 & 0 & -1 & 1 & 5 \\ 0 & 1 & 0 & -1 & 1 & 15 \\ 0 & 0 & 1 & -1 & 0 & 5 \\ 0 & 0 & -1 & 1 & 0 & -5 \end{array}\right] \begin{array}{c} \longrightarrow \\ \\ \\ R_4+R_3 \end{array}$$

$$\left[\begin{array}{rrrrr|r} 1 & 0 & 0 & -1 & 1 & 5 \\ 0 & 1 & 0 & -1 & 1 & 15 \\ 0 & 0 & 1 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

giving

$$f_1 = 5 + s - t, \quad f_2 = 15 + s - t, \quad f_3 = 5 + s, \quad f_4 = s \quad \text{and} \quad f_5 = t$$

for integers s, t (as we cannot have fractional trains). Moreover, as trains cannot go the other way, we immediately have

$$\begin{array}{rcrcr} f_1 \geq 0 & \Longrightarrow & 5 + s - t \geq 0 & \Longrightarrow & s - t \geq -5 \\ f_2 \geq 0 & \Longrightarrow & 15 + s - t \geq 0 & \Longrightarrow & s - t \geq -15 \\ f_3 \geq 0 & \Longrightarrow & 5 + s \geq 0 & \Longrightarrow & s \geq -5 \\ f_4 \geq 0 & \Longrightarrow & s \geq 0 & & \\ f_5 \geq 0 & \Longrightarrow & t \geq 0 & & \end{array}$$

so we have $s, t \geq 0$ and $s - t \geq -5$.

b) Assume the tracks from A to C and from D to A are closed. This forces $f_4 = f_5 = 0$. From our previous solution, we have that $s = t = 0$. Since $s - t = 0 \geq -5$, this is a valid solution. We have

$$f_1 = 5, \quad f_2 = 15, \quad f_3 = 5, \quad f_4 = 0 \quad \text{and} \quad f_5 = 0$$

Notice here we have a unique solution.

Application: Electrical Networks

Consider the following *electrical network* shown in Figure 48:

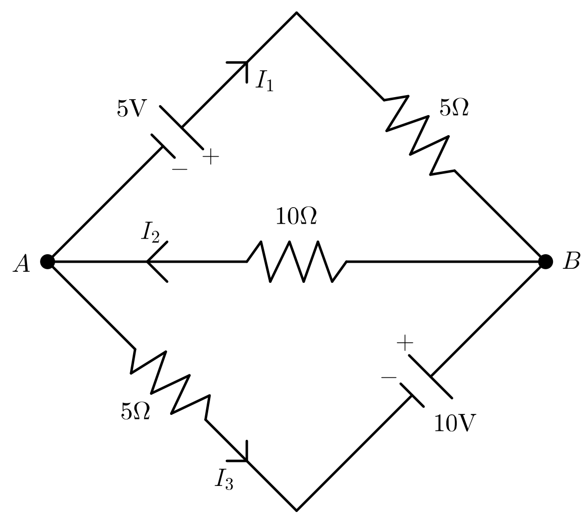


Figure 48: An electrical network

It consists of *voltage sources*, *resistors* and *wires*. A voltage source (often a battery) provides an *electromotive force* V measured in *volts*. This electromotive force moves electrons through the network along a wire at a rate we refer to as *current* I measured in *amperes* (or amps). The resistors (lightbulbs for example) are measured in *ohms* Ω, and serve to retard the current by slowing the flow of electrons. The intersection point between three or more wires is called a *node*. The nodes break the wires up into short paths between two nodes. Every such path can have a different current, and the arrow on each path is called a *reference direction*. Pictured here is a voltage source (left) and a resistor (right) between two nodes.

One remark about voltage sources. If a current passes through a battery supplying V volts from the "−" to the "+", then there is a voltage increase of V volts. If the current passes through the same battery from the "+" to the "−", then there is a voltage drop (decrease) of V volts.

Our aim is to compute the currents I_1, I_2 and I_3 in Figure 48. The following laws will be useful.

Ohm's Law The potential difference V across a resistor is given by $V = IR$, where I is the current and R is the resistance.

Note that the reference direction is important when using Ohm's Law. A current I travelling across a resistor of 10Ω in the reference direction will result in a voltage drop of $10I$ while the same current travelling across the same resistor against the reference direction will result in a voltage gain of $10I$.

Kirchoff's Laws

1. Conservation of Energy: Around any closed voltage loop in the network, the algebraic sum of voltage drops and voltage increases caused by resistors and voltage sources is zero.

2. Conservation of Charge: At each node, the total inflow of current equals the total outflow of current.

Kirchoff's Laws will be used to derive a system of equations that we can solve in order to find the currents. The Conservation of Energy requires using Ohm's Law. Returning to Figure 48, we can now solve for I_1, I_2 and I_3. Notice that there is an upper loop, and a lower loop. We may choose any orientation we like for either loop. Given the reference directions, we will use a clockwise orientation for the upper loop and a counterclockwise orientation for the lower loop. We will compute the voltage increases and drops as we move around both loops. Conservation of Energy says the voltage drops must equal the voltage gains around each loop.

For the upper loop, we can start at node A. Moving clockwise, we first have a voltage gain of 5 from the battery, then a voltage drop of $5I_1$ at the 5Ω resistor and a $10I_2$ voltage drop at the 10Ω resistor. Thus

$$5I_1 + 10I_2 = 5 \tag{15}$$

For the lower loop, we can again start at node A. Moving counterclockwise, we have a voltage drop of $5I_3$ followed by a voltage increase of 10 and finally a voltage drop of $10I_2$. We have

$$10I_2 + 5I_3 = 10 \tag{16}$$

Now, applying the Conservation of Charge to node A gives $I_1 + I_3 = I_2$ so we obtain

$$I_1 - I_2 + I_3 = 0 \tag{17}$$

Note that at node B we obtain the same equation, so including it would be redundant. Combining equations (15), (16) and (17) gives the system of equations

$$\begin{array}{rcrcrcr} I_1 & - & I_2 & + & I_3 & = & 0 \\ 5I_1 & + & 10I_2 & & & = & 5 \\ & & 10I_2 & + & 5I_3 & = & 10 \end{array}$$

Carrying the augmented matrix of this system to RREF,

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 5 & 10 & 0 & 5 \\ 0 & 10 & 5 & 10 \end{array}\right] \xrightarrow[R_2-5R_1]{} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 15 & -5 & 5 \\ 0 & 10 & 5 & 10 \end{array}\right] \xrightarrow[\frac{1}{5}R_2,\ \frac{1}{5}R_3]{} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 1 \\ 0 & 2 & 1 & 2 \end{array}\right] \xrightarrow[R_2-R_3]{}$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 2 & 1 & 2 \end{array}\right] \xrightarrow[R_3-2R_2]{R_1+R_2} \left[\begin{array}{rrr|r} 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 5 & 4 \end{array}\right] \xrightarrow[\frac{1}{5}R_3]{} \left[\begin{array}{rrr|r} 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 4/5 \end{array}\right] \xrightarrow[R_2+2R_3]{R_1+R_3}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & -1/5 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 1 & 4/5 \end{array}\right]$$

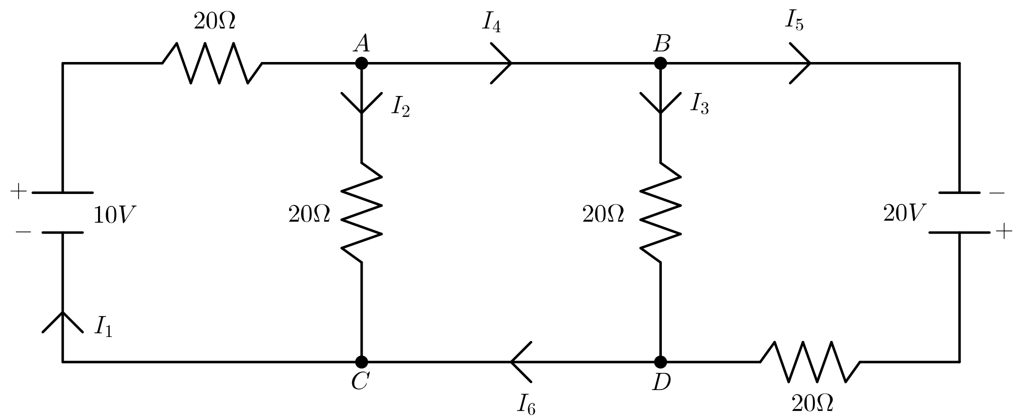
we see that $I_1 = -1/5$ amps, $I_2 = 3/5$ amps and $I_3 = 4/5$ amps. Notice that I_1 is negative. This simply means that our reference direction for I_1 in Figure 48 is incorrect and the current flows in the opposite direction there. Note that the reference directions may be assigned arbitrarily.

Note that there is actually a third loop in Figure 48: the loop that travels along the outside of the network. If we start at node A and travel clockwise around this loop, we first have a voltage increase of 5, then a voltage drop of $5I_1$, then another voltage drop of 10 (as we pass through the $10V$ battery from "+" to "−") and finally a voltage increase of $5I_3$ (as we pass through the 5Ω resistor in the opposite reference direction for I_3). As voltage increases equal voltage drops, we have $5 + 5I_3 = 5I_1 + 10$, or $5I_1 - 5I_3 = -5$. However, this is just Equation (16) subtracted from Equation (15). Including this equation in our above system of equations would only result in an extra row of zeros when we carried the resulting system of equations to RREF. This will be true in general, and shows that when computing current in an electrical network, we only need to consider the "smallest" loops.

Another note is that we chose to orient the upper loop in the clockwise direction and the lower loop in the counterclockwise direction. This was totally arbitrary (but made sense given the reference directions). We could have changed either of the directions. Of course, as we saw in the previous paragraph, we have to consider which way our orientation will cause the current to flow through a battery, and how to handle resistors if our orientation has us moving in the opposite direction of a reference direction.

One last thing to notice here is that since I_1 is negative, the current is actually flowing backwards through the $5V$ battery. This can happen in a poorly designed electrical network - the $10V$ battery is too strong and actually forces the current to travel through the $5V$ battery in the wrong direction. Too much current being forced through a battery in the wrong direction will lead to a fire.

Example 24.2. Find the currents in the following electrical network:



Solution. We begin by using the Conservation of Energy on each of the three smallest closed loops. Going clockwise around the left loop starting at A, we see a voltage drop of $20I_2$, a voltage gain of 10 and then a drop of $20I_1$. This gives

$$20I_1 + 20I_2 = 10 \quad \text{or} \quad 2I_1 + 2I_2 = 1$$

Traversing the middle loop clockwise starting at A, we have a voltage drop of $20I_3$ followed by a gain of $20I_2$ (note the we pass the resistor between A and C in the opposite direction of I_2). We obtain

$$20I_2 = 20I_3 \quad \text{or} \quad I_2 - I_3 = 0$$

Moving clockwise around the right loop starting at B, we observe a voltage gain of 20, followed by a drop of $20I_5$ and then a gain of $20I_3$ leading to

$$20I_5 = 20 + 20I_3 \quad \text{or} \quad I_3 - I_5 = -1$$

Next, we apply the Conservation of Charge to the nodes A, B, C and D (in that order) to obtain the equations

$$\begin{aligned} I_1 - I_2 - I_4 &= 0 \\ I_3 - I_4 + I_5 &= 0 \\ I_1 - I_2 - I_6 &= 0 \\ I_3 + I_5 - I_6 &= 0 \end{aligned}$$

Finally, we have constructed the system of equations

$$
\begin{array}{rcrcrcrcrcrcr}
2I_1 & + & 2I_2 & & & & & & & & & = & 1 \\
 & & I_2 & - & I_3 & & & & & & & = & 0 \\
 & & & & I_3 & & & - & I_5 & & & = & -1 \\
I_1 & - & I_2 & & & - & I_4 & & & & & = & 0 \\
 & & & & I_3 & - & I_4 & + & I_5 & & & = & 0 \\
I_1 & - & I_2 & & & & & & & - & I_6 & = & 0 \\
 & & & & I_3 & & & & I_5 & - & I_6 & = & 0
\end{array}
$$

Carrying the augmented matrix of this system to RREF, we have

$$
\left[\begin{array}{rrrrrr|r}
2 & 2 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & -1 \\
1 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 1 & -1 & 0
\end{array}\right]
\xrightarrow{R_1 \leftrightarrow R_4}
\left[\begin{array}{rrrrrr|r}
1 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & -1 \\
2 & 2 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 1 & -1 & 0
\end{array}\right]
\begin{array}{c} \longrightarrow \\ \\ \\ R_4-2R_1 \\ \\ R_6-R_1 \\ \\ \end{array}
$$

$$
\left[\begin{array}{rrrrrr|r}
1 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & -1 \\
0 & 4 & 0 & 2 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 1 & -1 & 0
\end{array}\right]
\begin{array}{c} R_1+R_2 \\ \longrightarrow \\ \\ R_4-4R_2 \\ \\ \\ \\ \end{array}
\left[\begin{array}{rrrrrr|r}
1 & 0 & -1 & -1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & -1 \\
0 & 0 & 4 & 2 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 1 & -1 & 0
\end{array}\right]
\begin{array}{c} R_1+R_3 \\ R_2+R_3 \\ \longrightarrow \\ R_4-4R_3 \\ R_5-R_3 \\ \\ R_7-R_3 \end{array}
$$

$$
\left[\begin{array}{rrrrrr|r}
1 & 0 & 0 & -1 & -1 & 0 & -1 \\
0 & 1 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 2 & 4 & 0 & 5 \\
0 & 0 & 0 & -1 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 2 & -1 & 1
\end{array}\right]
\begin{array}{c} \longrightarrow \\ \\ \\ \\ R_4 \leftrightarrow R_6 \\ \\ \\ \end{array}
\left[\begin{array}{rrrrrr|r}
1 & 0 & 0 & -1 & -1 & 0 & -1 \\
0 & 1 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & 0 & 1 \\
0 & 0 & 0 & 2 & 4 & 0 & 5 \\
0 & 0 & 0 & 0 & 2 & -1 & 1
\end{array}\right]
\begin{array}{c} R_1+R_4 \\ \longrightarrow \\ \\ \\ R_5+R_4 \\ R_6-2R_4 \\ \\ \end{array}
$$

$$\left[\begin{array}{cccccc|c} 1&0&0&0&-1&-1&-1\\ 0&1&0&0&-1&0&-1\\ 0&0&1&0&-1&0&-1\\ 0&0&0&1&0&-1&0\\ 0&0&0&0&2&-1&1\\ 0&0&0&0&4&2&5\\ 0&0&0&0&2&-1&1 \end{array}\right] \begin{array}{c} \longrightarrow\\ \\ \\ \\ \frac{1}{2}R_4\\ \frac{1}{4}R_5\\ \frac{1}{2}R_7 \end{array} \left[\begin{array}{cccccc|c} 1&0&0&0&-1&-1&-1\\ 0&1&0&0&-1&0&-1\\ 0&0&1&0&-1&0&-1\\ 0&0&0&1&0&-1&0\\ 0&0&0&0&1&-1/2&1/2\\ 0&0&0&0&1&1/2&5/4\\ 0&0&0&0&1&-1/2&1/2 \end{array}\right] \begin{array}{c} R_1+R_5\\ R_2+R_5\\ R_3+R_5\\ \longrightarrow\\ \\ R_6-R_5\\ R_7-R_5 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1&0&0&0&0&-3/2&-1/2\\ 0&1&0&0&0&-1/2&-1/2\\ 0&0&1&0&0&-1/2&-1/2\\ 0&0&0&1&0&-1&0\\ 0&0&0&0&1&-1/2&1/2\\ 0&0&0&0&0&1&3/4\\ 0&0&0&0&0&0&0 \end{array}\right] \begin{array}{c} R_1+\frac{3}{2}R_6\\ R_2+\frac{1}{2}R_6\\ R_3+\frac{1}{2}R_6\\ R_4+R_6\\ R_5+\frac{1}{2}R_6\\ \longrightarrow\\ \\ \end{array} \left[\begin{array}{cccccc|c} 1&0&0&0&0&0&5/8\\ 0&1&0&0&0&0&-1/8\\ 0&0&1&0&0&0&-1/8\\ 0&0&0&1&0&0&3/4\\ 0&0&0&0&1&0&7/8\\ 0&0&0&0&0&1&3/4\\ 0&0&0&0&0&0&0 \end{array}\right]$$

Finally, we see

$$I_1 = \frac{5}{8}\text{amps}, \quad I_2 = -\frac{1}{8}\text{amps}, \quad I_3 = -\frac{1}{8}\text{amps},$$

$$I_4 = \frac{3}{4}\text{amps}, \quad I_5 = \frac{7}{8}\text{amps}, \qquad I_6 = \frac{3}{4}\text{amps}$$

In particular, the reference arrows for I_2 and I_3 are pointing in the wrong direction.