Dot Products

Next we learn some vector operations that will be useful to us in doing some geometry. In many ways, vector algebra is the right language for geometry, particularly if we're using functions. In a way, vector algebra is a language and we're using it to express things we've known since childhood. Having a notation for these things will make them more straightforward.

A dot product is a way of multiplying two vectors to get a number, or scalar. Algebraically, suppose $\mathbf{A} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3 \rangle$. We find the *dot product* $\mathbf{A} \cdot \mathbf{B}$ by multiplying the first component of $\mathbf{A}$ by the first component of $\mathbf{B}$, the second component of $\mathbf{A}$ by the second component of $\mathbf{B}$, and so on, and then adding together all these products. So for our sample vectors, $\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$. If our vectors have N components, the definition of the dot product becomes:

$$\mathbf{A} \cdot \mathbf{B} = \sum_{i=1}^{N} a_i b_i.$$

It is very important to remember that $\mathbf{A} \cdot \mathbf{B}$ is a scalar, not a vector. Also, when writing a dot product we always put a dot symbol between the two vectors to indicate what kind of product we're calculating.

What is it good for? The answer to this question will be clearer after we see a geometric description of the dot product.

Geometrically, the dot product of $\mathbf{A}$ and $\mathbf{B}$ equals the length of $\mathbf{A}$ times the length of $\mathbf{B}$ times the cosine of the angle between them:

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos(\theta).$$

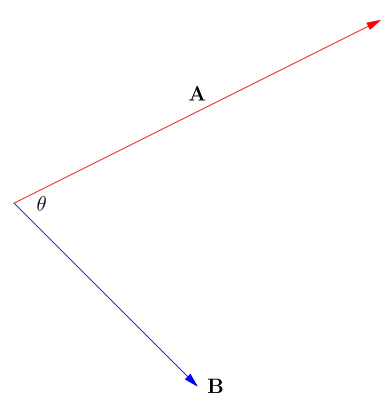

Figure 1: $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos(\theta)$.

This may seem complicated and artificial at first, but we'll find that the dot product gives us useful information about angles and lengths simultaneously. If we've described our vectors using components, the dot product is also easy to calculate.

Equivalence of Descriptions of the Dot Product

We now have two descriptions of the dot product:

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos(\theta) \qquad \text{and} \qquad \mathbf{A} \cdot \mathbf{B} = \sum_{i=1}^{N} a_i b_i.$$

In mathematics, one tries to justify every statement by proving theorems. The first theorem in 18.02 is to verify that if the dot product is defined to be $\mathbf{A} \cdot \mathbf{B} = \sum_{i=1}^{N} a_i b_i$ then it's also true that $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos(\theta)$.

We start by multiplying a vector times itself to gain understanding of the geometric definition:

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 \cos(0) = |\mathbf{A}|^2.$$

From the definition of the dot product we get:

$$\mathbf{A} \cdot \mathbf{A} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{A}|^2.$$

The two definitions are equivalent if $\mathbf{A}$ and $\mathbf{B}$ are the same vector.

If $\mathbf{A}$ and $\mathbf{B}$ are different vectors, we can use the law of cosines to show that our geometric description of the dot product of two different vectors is equivalent to its algebraic definition. You may recall that the law of cosines tells you the length of the third side of a triangle given the length of the other two sides and the angle between them.

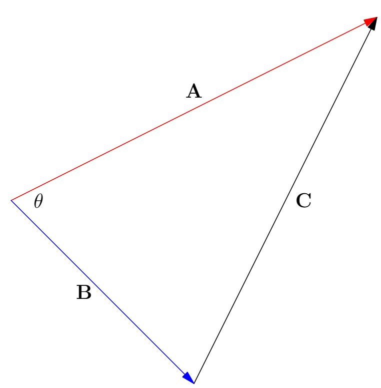

Figure 2: The law of cosines describes the length of $\mathbf{C}$ in terms of $|\mathbf{A}|$, $|\mathbf{B}|$ and θ.

In terms of vectors, the two known sides of our triangle are formed by $\mathbf{A}$ and $\mathbf{B}$. The third side is described by $\mathbf{C} = \mathbf{A} - \mathbf{B}$. The law of cosines then tells us that:

$$|\mathbf{C}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}|\cos\theta.$$

(If you haven't seen this before, then consider what you are about to see to be a proof of the law of cosines based on the assumption that our two descriptions of the dot product are equivalent. If you have seen the law of cosines before, it's the other way around.)

How is the law of cosines related to the dot product?

$$\begin{aligned}
|\mathbf{C}|^2 &= \mathbf{C}\cdot\mathbf{C} \\
&= (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) \\
&= \mathbf{A}\cdot\mathbf{A}-\mathbf{A}\cdot\mathbf{B}-\mathbf{B}\cdot\mathbf{A}+\mathbf{B}\cdot\mathbf{B} \\
|\mathbf{C}|^2 &= |\mathbf{A}|^2+|\mathbf{B}|^2-2\mathbf{A}\cdot\mathbf{B}
\end{aligned}$$

Are we allowed to expand $(\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B})$ in this way? Is it true that $\mathbf{A}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{A}$? Yes; it is possible to prove from the definition of the dot product that commuting, factoring and expanding work with dot products the same way they do with scalar products. (This is where we use the definition of the dot product in this proof.)

Comparing this formula for the length of $\mathbf{C}$ with the one given by the law of cosines, we see that we must have $2\mathbf{A}\cdot\mathbf{B}=2|\mathbf{A}||\mathbf{B}|\cos\theta$, and so we conclude that:

$$\mathbf{A}\cdot\mathbf{B}=|\mathbf{A}||\mathbf{B}|\cos(\theta).$$

Now we have either used the law of cosines to prove that our algebraic and geometric descriptions of the dot product are equivalent, or we have proven the law of cosines based on the assumption that those descriptions are equivalent. A mathematician wold say that the law of cosines is logically equivalent to the statement $\mathbf{A}\cdot\mathbf{B}=|\mathbf{A}||\mathbf{B}|\cos\theta$.