SE 212 Fall 2022
Summary of Set Theory

1 Axioms

- Types as sets (if $\exists x \bullet x \in B$)
  $\forall x : B \bullet P(x) \quad \Leftrightarrow \quad \forall x \bullet x \in B \Rightarrow P(x)$
  $\exists x : B \bullet P(x) \quad \Leftrightarrow \quad \exists x \bullet x \in B \land P(x)$
- Set Comprehension
  $x \in \{y : S \mid P(y)\} \quad \Leftrightarrow \quad x \in S \land P(x)$
  $x \in \{t(a, b, \dots) \bullet a : S, b : R, \dots \mid P(a, b, \dots)\}$
  $\Leftrightarrow$
  $\exists a, b, \dots \bullet a \in S \land b \in R \land \dots \land x = t(a, b, \dots) \land P(a, b, \dots)$
- Empty Set
  $\forall x \bullet \neg(x \in \emptyset)$
- Universal Set
  $\forall x \bullet x \in U$
- Set Equality
  $D = B \Leftrightarrow (\forall x \bullet x \in D \Leftrightarrow x \in B)$
- Subset
  $D \subseteq B \Leftrightarrow (\forall x \bullet x \in D \Rightarrow x \in B)$
- Proper Subset
  $D \subset B \Leftrightarrow D \subseteq B \land \neg(D = B)$
- Power Set
  $\mathbb{P}(D) = \{B \mid B \subseteq D\}$
- Set Union
  $D \cup B = \{x \mid x \in D \lor x \in B\}$
- Set Intersection
  $D \cap B = \{x \mid x \in D \land x \in B\}$
- Absolute Complement
  $compl(A) = \{x \mid \neg(x \in A)\}$
  (difference from the universal set)
- Set Difference
  $D - B = \{x \mid x \in D \land \neg(x \in B)\}$
  (relative Complement)
- Set Product
  $D * B = \{(d, b) \mid d \in D \land b \in B\}$

Note that if we have $x \in D$, we can expand the definition of D and apply the set comprehension axiom in one step.

Relations

- Domain
  $\text{dom}(R) = \{x \mid \exists y \bullet (x, y) \in R\}$
- Range
  $\text{ran}(R) = \{y \mid \exists x \bullet (x, y) \in R\}$
- Inverse
  $R^{\sim} = \{(b, a) \mid (a, b) \in R\}$
- Identity
  $\text{id}(A) = \{(a, a) \mid a \in A\}$
  $\text{id}(A) = \{(a, b) \mid a \in A \land b \in A \land (a = b)\}$
- Relational Image
  $R(\!| S |\!) = \{y \mid \exists x \bullet (x, y) \in R \land x \in S\}$
- Relational Composition (where $R : D \leftrightarrow B, S : B \leftrightarrow C$)
  $S \circ R = R; S = \{(a, c) \mid \exists b \bullet (a, b) \in R \land (b, c) \in S\}$
- Domain Restriction
  $S \lhd R = \{(a, b) \mid (a, b) \in R \land a \in S\}$
- Domain Subtraction
  $S \ntriangleleft R = \{(a, b) \mid (a, b) \in R \land \neg(a \in S)\}$
- Range Restriction
  $R \rhd S = \{(a, b) \mid (a, b) \in R \land b \in S\}$
- Range Subtraction
  $R \ntriangleright S = \{(a, b) \mid (a, b) \in R \land \neg(b \in S)\}$
- Relational Overriding
  $R \oplus S = (\text{dom}(S) \ntriangleleft R) \cup S$
  $R \oplus S = \{(a, b) \mid (a, b) \in R \land \neg(a \in \text{dom}(S)) \lor (a, b) \in S\}$

Name	**Notation**	**Name**	**Notation**
Relation	$R : D \leftrightarrow B$		
Total Fcn	$f : D \rightarrow B$	Non-total Fcn	$f : D \nrightarrow B$
Total Injective (1-to-1) Fcn	$f : D \rightarrowtail B$	Non-total Injective (1-to-1) Fcn	$f : D \rightarrowtail\!\!\!\!\!+ B$
Total Surjective (onto) Fcn	$f : D \twoheadrightarrow B$	Non-total Surjective (onto) Fcn	$f : D \twoheadrightarrow\!\!\!\!\!+ B$
Total Bijective Fcn	$f : D \rightarrowtail\!\!\!\!\!\twoheadrightarrow B$	Non-total Bijective Fcn	$f : D \rightarrowtail\!\!\!\!\!\twoheadrightarrow\!\!\!\!\!+ B$

2 Derived Laws

- The empty subset is a subset of every set.

  $\emptyset \subseteq D$

- Every set is a subset of itself.

  $D \subseteq D$

- Subset is transitive.

  $D \subseteq B \land B \subseteq C \Rightarrow D \subseteq C$

- Power Set

  $S \in \mathbb{P}(Q) \Leftrightarrow (\forall x \bullet x \in S \Rightarrow x \in Q)$

- Set Equality Properties

  $A = A$
  $(D = B) \quad \Leftrightarrow \quad D \subseteq B \land B \subseteq D$

- Commutative

  $D \cap B = B \cap D$
  $D \cup B = B \cup D$

- Associative

  $(D \cap B) \cap C = D \cap (B \cap C)$
  $(D \cup B) \cup C = D \cup (B \cup C)$

- Distributive

  $(D \cup B) \cap C = (D \cap C) \cup (B \cap C)$
  $(D \cap B) \cup C = (D \cup C) \cap (B \cup C)$

- Complement Laws

  $\mathit{compl}(D \cap B) = \mathit{compl}(D) \cup \mathit{compl}(B)$
  $\mathit{compl}(D \cup B) = \mathit{compl}(D) \cap \mathit{compl}(B)$
  $\mathit{compl}(\mathit{compl}(D)) = D$

- Empty Set Identities

  $D \cap \emptyset = \emptyset$
  $D \cup \emptyset = D$
  $D - \emptyset = D$
  $\emptyset - D = \emptyset$
  $\mathit{compl}(\emptyset) = U$

- Universal Set Identities

  $D \cap U = D$
  $D \cup U = U$
  $D - U = \emptyset$
  $U - D = \mathit{compl}(D)$
  $\mathit{compl}(U) = \emptyset$
  $U = D \cup \mathit{compl}(D)$

- Intersection is Subset

  $D \cap B \subseteq D$

- Subset of Union

  $D \subseteq D \cup B$

- Domain Properties $(S, T : D \leftrightarrow B)$

  $\mathrm{dom}(S) \subseteq D$
  $\mathrm{dom}\ (S \cup T) = \mathrm{dom}(S) \cup \mathrm{dom}(T)$
  $\mathrm{dom}\ (S \cap T) \subseteq \mathrm{dom}(S) \cap \mathrm{dom}(T)$

- Range Properties $(S, T : D \leftrightarrow B)$

  $\mathrm{ran}(S) \subseteq B$
  $\mathrm{ran}\ (S \cup T) = \mathrm{ran}(S) \cup \mathrm{ran}(T)$
  $\mathrm{ran}\ (S \cap T) \subseteq \mathrm{ran}(S) \cap \mathrm{ran}(T)$

- Inverse Properties

  $(R^\sim)^\sim = R$
  $\mathrm{dom}\ (R^\sim) = \mathrm{ran}(R)$
  $\mathrm{ran}\ (R^\sim) = \mathrm{dom}(R)$

- Relational Composition Properties
  $(R : D \leftrightarrow B, S : B \leftrightarrow C, T : C \leftrightarrow D)$

  $R; (S; T) = (R; S); T \qquad (T \circ S) \circ R = T \circ (S \circ R)$
  $(R; S)^\sim = S^\sim; R^\sim$
  $\mathrm{id}(\mathrm{dom}(R)) \subseteq R; R^\sim$
  $\mathrm{id}(\mathrm{ran}(R)) \subseteq R^\sim; R$
  $(\mathrm{id}(D); R = R$
  $R; (\mathrm{id}(B)) = R$

- Relational Image Properties $(R : X \leftrightarrow Y, D, B : X)$

  $R(\!| D \cup B |\!) = R(\!| D |\!) \cup R(\!| B |\!)$
  $R(\!| D \cap B |\!) \subseteq R(\!| D |\!) \cap R(\!| B |\!)$
  $R(\!| \mathrm{dom}\ (R) |\!) = \mathrm{ran}\ (R)$

- Domain Restriction Properties $(R : X \leftrightarrow Y, D, B : X)$

  $D \lhd (B \lhd R) = (D \cap B) \lhd R$
  $D ⩤ (B ⩤ R) = (D \cup B) ⩤ R$
  $(D \lhd R) \cup (D ⩤ R) = R$

- Range Restriction Properties $(R : X \leftrightarrow Y, D, B : Y)$

  $(R \rhd D) \rhd B = R \rhd (D \cap B)$
  $(R ⩥ D) ⩥ B = R ⩥ (D \cup B)$
  $(R \rhd D) \cup (R ⩥ D) = R$

- Relational Overriding Properties $(R, S, T : X \leftrightarrow Y)$

  $R \oplus R = R$
  $R \oplus (S \oplus T) = (R \oplus S) \oplus T$